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A parametric representation of ellipses and ellipsoids

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Abstract—The parametric approach to the specification of geometric form is particularly well suited to the needs of structural geologists. This note illustrates its application to ellipses and ellipsoids which may represent stress, strain, or other rank-2 tensor phenomena in rocks. Compared to traditional functional expressions, parametric equations are simpler, faster to compute, and more meaningful in terms of physical parameters.

INTRODUCTION

ELLIPSES and ellipsoids are fundamental to the geometric description of geological deformation. The implicit equation for an ellipse with semi-axes a and b parallel to x and y reference axes is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \quad (1)$$

However, when the axes are oblique to the reference directions, the equation becomes unwieldy,

$$(c^2 + e^2)x^2 + 2(cd + ef)xy + (d^2 + f^2)y^2 = 1, \quad (2)$$

where c, d, e, and f are inverse transformation parameters determining the ellipse's shape and orientation (e.g. Twiss & Moores 1992, p. 294). It is possible to specify the deformation state using a forward transformation tensor, D, but it is difficult to get any feel for the significance of the numbers involved, except in a special case such as simple shear,

$$D = \begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

and even that becomes obscure when the shear plane or shear direction are oblique to the reference axes. The purpose of this note is to present a remarkably simple parametric approach to the specification of stress, strain, fabric, and object ellipses and ellipsoids in a general reference frame.

PARAMETRIC EQUATIONS

Lines

Readers will be familiar with the explicit equation of a straight line

$$y = mx + c \quad (4)$$

where (x, y) are abscissa and ordinate, m is the slope, and c is the y-axis intercept. This equation expresses the dependent variable y as a function of the independent variable x. It is not very convenient when the problem is to connect two given points,

$$\mathbf{p}_1 = (x_1,y_1) \quad (5)$$

$$\mathbf{p}_2 = (x_2,y_2), \quad (6)$$

and it breaks down when the slope is vertical. An implicit function such as the equation of a unit circle,

$$x^2 + y^2 = 1 \quad (7)$$

must be converted to explicit form in practice,

$$y = \sqrt{(1 - x^2)} \quad (8)$$

and the problem of vertical slope recurs. The parametric equation of a straight line joining two points $\mathbf{p}_1$ and $\mathbf{p}_2$ is

$$\mathbf{p} = s\mathbf{p}_1 + t\mathbf{p}_2, \quad (9)$$

where $s{:}t$ is a proportional division of the line such that

$$s + t = 1. \quad (10)$$

For example, if s is set to 30% and t to 70% then the point

$$\mathbf{p} = 0.3\,\mathbf{p}_1 + 0.7\,\mathbf{p}_2 \quad (11)$$

lies on the line. You could think of point $\mathbf{p}$ as being

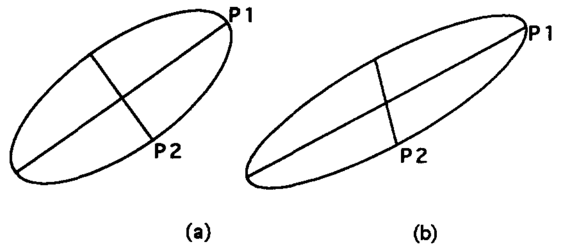

Fig. 1. Parametric representation of an ellipse. P1 and P2 are the given radii. (a) P1 and P2 are orthogonal and form the axes of the ellipse. (b) P1 and P2 are non-orthogonal. P2 is parallel to the ellipse's tangent at the tip of P1 and vice versa.

"composed of" 30% of the co-ordinates of $\mathbf{p}_1$ and 70% of the co-ordinates of $\mathbf{p}_2$. If $t < 0$ or $t > 1$ then the point is on the line but external to the segment joining $\mathbf{p}_1$ to $\mathbf{p}_2$. The point

$$\mathbf{p} = 0.5\,\mathbf{p}_1 + 0.4\,\mathbf{p}_2 \tag{12}$$

is not on the line, however, because s and t do not sum to unity in this case. Each cartesian co-ordinate may be substituted for $\mathbf{p}$ to form simultaneous equations,

$$x = sx_1 + tx_2 \tag{13}$$

$$y = sy_1 + ty_2. \tag{14}$$

Although two parameters, s and t are used in this paper for the sake of presentation, there is only one independent parameter because of equation (10). For a conventional treatment of parametric equations in one parameter see, for example Foley *et al.* (1990). The parameters may be thought of as time and time remaining, for example; if one is watching a line being drawn, then the pen is at a definable point at each instant in time, even if the line's slope is vertical. The equation holds equally in two or three dimensions,

$$x = sx_1 + tx_2 \tag{15}$$

$$y = sy_1 + ty_2 \tag{16}$$

$$z = sz_1 + tz_2. \tag{17}$$

It is important to note the values of s and t are common. The complete line can be drawn at any desired level of resolution by repeating equation (9) for sufficiently small increments of t (e.g. $t = 0.001, 0.002, 0.003$, etc.).

Ellipses

A similar approach may be taken to the construction of an ellipse centered at the origin, given two radius vectors $\mathbf{p}_1$ and $\mathbf{p}_2$:

$$\mathbf{p} = \sqrt{s}\mathbf{p}_1 + \sqrt{t}\mathbf{p}_2, \tag{18}$$

where t is confined to the range [0,1]. Remember that s is simply 1-t, not to be confused with an ellipse axis. If $\mathbf{p}_1$ and $\mathbf{p}_2$ are perpendicular then they define the semi-axes of the ellipse (Fig. 1a)—otherwise they are a special pair of radii such that $\mathbf{p}_1$ is parallel to the tangent at the tip of $\mathbf{p}_2$ and vice versa (Fig. 1b). Each combination of signs $\pm\sqrt{s}$ and $\pm\sqrt{t}$ gives one quadrant of the ellipse. If the ellipse is centered at a point $\mathbf{p}_0$, other than the origin, its equation is simply

$$\mathbf{p} = \sqrt{s}\mathbf{p}_1 + \sqrt{t}\mathbf{p}_2 + \mathbf{p}_0. \tag{19}$$

Note that equal increments of t do not yield equal arc lengths of the ellipse; rather, points are closer in regions of higher curvature, which is efficient for computerized drafting. Equation (19) compares favorably with equation (2), not only because there are fewer terms, but also because the shape and orientation of the ellipse can be visualized from the values in vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. Furthermore, if the ellipse is deformed such that the center moves from $\mathbf{p}_0$ to $\mathbf{p}_0'$ whilst vector $\mathbf{p}_1$ transforms to $\mathbf{p}_1'$ and $\mathbf{p}_2$ to $\mathbf{p}_2'$, then the deformed ellipse is given by

$$\mathbf{p}' = \sqrt{s}\mathbf{p}_1' + \sqrt{t}\mathbf{p}_2' + \mathbf{p}_0'. \tag{20}$$

which is a remarkable simplification for the basic equations of R_f/ϕ and Fry analysis (e.g. Marshak & Mitra 1988, p. 352). Note that $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are not the semi-axes of the deformed ellipse unless except in special cases. Square roots are not quick to calculate on a microcomputer, but their values may be pre-processed and stored in look-up tables. Speed is not a concern for most applications but becomes significant when simulating deformation in real time, as in the author's "Strain Grid" program, which redraws 100 deformed ellipses while the computer's mouse is in motion. Conventional calculation using equation (2) would be too slow in this case.

Planes and ellipsoids

The equation of a plane is given by

$$\mathbf{p} = r\mathbf{p}_1 + s\mathbf{p}_2 + t\mathbf{p}_3, \tag{21}$$

where r, s, and t sum to unity. Thus the point $\mathbf{p}$ lies on the plane if

$$r{:}s{:}t = 20\%{:}50\%{:}30\%, \tag{22}$$

for example, but the point (0.4 $\mathbf{p}_1$, 0.4 $\mathbf{p}_2$, 0.4 $\mathbf{p}_3$) does not because the parameter sum is 1.2

Similarly, the equation of an ellipsoid, given three radii $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, may be written

$$\mathbf{p} = \sqrt{r}\mathbf{p}_1 + \sqrt{s}\mathbf{p}_2 + \sqrt{t}\mathbf{p}_3 + \mathbf{p}_0. \tag{23}$$

(if off-origin, $\mathbf{p}_0$ denotes the center). The eight combinations of signs of $\pm\sqrt{r}$, $\pm\sqrt{s}$, and $\pm\sqrt{t}$ give the eight octants of the ellipsoid. This equation is short for three simultaneous co-ordinate equations

$$x = \sqrt{r}x_1 + \sqrt{s}x_2 + \sqrt{t}x_3 + x_0 \tag{24}$$

$$y = \sqrt{r}y_1 + \sqrt{s}y_2 + \sqrt{t}y_3 + y_0 \tag{25}$$

$$z = \sqrt{r}z_1 + \sqrt{s}z_2 + \sqrt{t}z_3 + z_0. \tag{26}$$

Here we see how powerful and succinct the parametric approach is when compared with the explicit functional or tensor transformation approach.

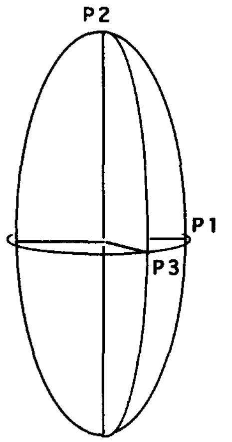

Fig. 2. Parametric equations define the three principal sections of an ellipsoid in a very simple formulation (see text).

Projection of ellipsoids

To convey a meaningful impression of stress or strain parameters, it is necessary to present data graphically, which requires a projected view in three dimensions. Again, the parametric approach is astonishingly simple. We are given the origin $\mathbf{p}_0$ and the unit vectors **x**, **y** and **z** in the co-ordinate directions. For example, let the origin be at $\mathbf{p}_0 = (0,0,0)$ and let the unit vectors project onto the plane of the diagram at

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \mathbf{z} = \begin{bmatrix} 0.25 \\ -0.125 \end{bmatrix}. \quad (27)$$

Then the equations of the three principal elliptical sections of the ellipsoid projected onto the **xy**-plane are

$$\mathbf{p} = \sqrt{s_1}\mathbf{x} + \sqrt{t_1}\mathbf{y} \quad (28)$$

$$\mathbf{p} = \sqrt{s_2}\mathbf{y} + \sqrt{t_2}\mathbf{z} \quad (29)$$

$$\mathbf{p} = \sqrt{s_3}\mathbf{z} + \sqrt{t_3}\mathbf{x}. \quad (30)$$

(Fig. 2). Note that **x**, **y**, and **z** are vectors with *projected* co-ordinates as shown in equation (27). Equations (28)–(30) thus stand for six equations, two for the plane co-ordinates of each ellipse.

CONCLUSIONS

Parametric representation of geometric form is ideally suited to geological applications where curves commonly have locally infinite slopes. The parametric form of an ellipse and ellipsoid presented in this note is dramatically simpler than any other type of representation, especially when the ellipse or ellipsoid is off origin and oriented oblique to some or all reference axes. The method is particularly elegant when three-dimensional results are projected onto the plane for graphical presentation. The method has already been applied to stress and strain analysis in De Paor (1990) and has further applications to forms other than ellipses and ellipsoids. For example,

$$\mathbf{p} = s^n\mathbf{p}_1 + t^n\mathbf{p}_2. \quad (31)$$

is identical to equation (18) when $n = 0.5$ but describes sub- and super-ellipse shapes (Gardner 1965, Lisle 1981, De Paor 1988) when $0 < n < 1$. It is also the basis for the mathematical manipulation of Bézier polynomials which have wide-ranging implications for structural modelling (work in progress).

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REFERENCES

De Paor, D. G. 1988. R_f/ϕ_f strain analysis using an orientation net. *J. Struct. Geol.* **10**, 323–333.

De Paor, D. G. 1990. The theory of shear stress and shear strain on planes inclined to the principal directions. *J. Struct. Geol.* **12**, 923–927.

Foly, J. D., van Dam, A., Feiner, S. K. & Hughes, J. F. 1990. *Computer Graphics: Principles and Practice*. (2nd edn). Addison-Wesley, Reading, Massachusetts.

Gardner, M. 1965. Mathematical games. *Scient. Am.* **213**(3), 222–232.

Lisle, R. J. 1981. The exact shape of deformed pebbles. *J. Struct. Geol.* **3**, 189 (abstract).

Marshak, S. & Mitra, G. 1988. *Basic Methods of Structural Geology*. Prentice-Hall, Englewood Cliffs, New Jersey.

Twiss, R. J. & Moores, E. M. 1992. *Structural Geology*. (1st edn). Freeman, San Francisco, California.